SHARP LOWER BOUNDS FOR THE MOMENTS OF SYMMETRIC PROBABILITY DENSITY FUNCTIONS

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Abstract

This work provides sharp lower bounds for the moments of some classes of symmetric probability density functions with the same second moment. The bounds are sharp and can not be improved. It proven that the distribution which has as moments the bounds is the uniform distribution.

Running tittle: Lower bounds for moments.

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1. Introduction

Let us consider the probability density functions (p.d.f.'s) $f : \Re \rightarrow [0, \infty)$ with the following properties:

P1. f is symmetric about zero.

P2. The restriction of f into $[0, \infty)$ is a nonincreasing function.

P3. The corresponding distribution function $F(x) = \int_{-\infty}^{x} f(x)dx$ is continuous for all x.

P4. $m_{2n} = \int x^{2n} f(x)dx < \infty$ for all n.

P5. $m_2 = m > 0$ where m is a fixed number.

Let us define the following set $\Psi_m^{(1)}$ of p.d.f.'s.

$$\Psi_m^{(1)} = \{f : f \text{ is a p.d.f. which satisfies the properties P1 through P5}\}.$$

We shall produce lower bounds for the moments of the p.d.f's in $\Psi_m^{(1)}$ and we shall prove that the bounds are the moments of the p.d.f. which correspond to the uniform distribution. If we put the extra assumption of unimodality at the class $\Psi_m^{(1)}$ the bounds can not be improved. For the wide class of all the p.d.f.'s with all the moments finite and the same first moment, the lower bounds are obtained by the degenerate p.d.f. We shall make use of the Gauss-Winckler inequality (Faber [2]) which states that if $f : (0, \infty) \to [0, \infty)$ is a nonincreasing function such that $\int_0^\infty f(x)dx = 1$, then for $m_k = \int_0^\infty x^k f(x)dx$, holds

$$[(1 + b)m_b]^{c-a} \leq [(1 + a)m_a]^{c-b}[(1 + c)m_c]^{b-a} \text{ whenever } 0 \leq a < b < c. \quad (1)$$

Avkhadiev [1] gave a simpler proof of (1). The simpler version of (1)

$$[(1 + a)m_a]^{1/a} \leq [(1 + c)m_c]^{1/c} \text{ for } 0 < a < c, \quad (2)$$

also appears in Kendall and Stuard [5] p. 92, under more strict assumptions of f.

We shall use the notation ν_k for the absolute moments $\nu_k = \int |x|^k f(x)dx$. The symbol $\diamondsuit$ indicates the end of a proof.

2. Some preliminary results

Let us first produce a more convenient version of (1).

Lemma 1. *For any f which satisfies assumptions of Gauss-Winckler inequality, the following inequality holds.*

$$\frac{(1 + (a + c)/2)^2}{(1 + a)(1 + c)} m^2_{(a+c)/2} \leq m_a m_c \leq \frac{(1 + a + c)}{(1 + a)(1 + c)} m_{a+c}. \quad (3)$$

Proof. Let $a \leq c$. Using (2) we obtain

$$(1 + a)m_a(1 + c)m_c < [(1 + c)m_c]^{a/c}(1 + c)m_c = [(1 + c)m_c]^{(a+c)/c}. \quad (4)$$

Applying once again inequality (2) for c and $a + c$ we obtain

$$[(1 + c)m_c]^{1/c} \leq (1 + a + c)m_{a+c}]^{1/(a+c)}. \quad (5)$$

From (4) and (5) we obtain

$$(1+a)m_a(1+c)m_c \leq (1+a+c)m_{a+c},$$

from which we conclude

$$m_a m_c \leq \frac{(1+a+c)}{(1+a)(1+c)} m_{a+c}. \tag{6}$$

Using $b = (a+c)/2$, in (1) we obtain

$$[(1+(a+c)/2)m_{(a+c)/2}]^{c-a} \leq [(1+a)m_a]^{(c-a)/2}[(1+c)m_c]^{(c-a)/2}.$$

After some simplifications we have

$$(1+(a+c)/2)^2 m^2_{(a+c)/2} \leq (1+a)m_a(1+c)m_c$$

or

$$\frac{(1+(a+c)/2)^2}{(1+a)(1+c)} m^2_{(a+c)/2} \leq m_a m_c. \tag{7}$$

Combining (6) and (7) we yield inequality (3). ♢

Inequality (7) appears in Hardy et. al. [4] p.166, but the proof here is simpler.

If f is a p.d.f from Ψ_m then the function $2fI_{(0,\infty)}$ satisfies the assumptions of Gauss-Winckler inequality. Thus inequalities (1), (2) and (3) hold for p.d.f's from $\Psi_m^{(1)}$.

We shall also make use of the inequality

$$\nu_r \nu_s < \nu_{r+s}, \tag{8}$$

which is a simple application of Lyapunov's inequality (Petrov [6] p.7). Moreover, with the use of the generalized version of Lyapunov's inequality

$$\nu_t^{r-s} \leq \nu_r^{s-t} \nu_t^{r-s} \text{ for } r \geq t \geq s, \tag{9}$$

we can produce a lower bound for the product $\nu_r \nu_s$ in (10). Indeed, using $t = (r+s)/2$ in (9), and working as in Lemma 1, we obtain

$$\nu_{(r+s)/2}^2 \leq \nu_r \nu_s. \tag{10}$$

Thus from (8) and (10) we yield

$$\nu_{(r+s)/2}^2 < \nu_r \nu_s < \nu_{r+s}.$$

3. Main Results

Proposition 1. *If f is a p.d.f. from $\Psi_m^{(1)}$ then*

$$m_{2n} \geq \frac{1}{(2n+1)(3m)^n}. \tag{11}$$

Proof. From inequality (6) for $a = 2n, a = 2n-2, ..., a = 4$ and $b = 2$ we obtain successively

$$m_{2n} \geq \frac{(2n-1)}{2n+1} m_{2n-2} 3m$$

$$m_{2n-2} \geq \frac{(2n-3)}{2n-1} m_{2n-4} 3m$$

$$\vdots$$

$$m_4 \geq \frac{(3m)}{5} 3m$$

Now easily we can obtain inequality (11). $\diamondsuit$

The question which arises is whether there is any distribution f^* with even moments m^*_{2n} the lower bounds

$$m^*_{2n} = \frac{1}{2n+1}(3m)^n. \tag{12}$$

Lemma 2. *The only symmetric p.d.f. which has as even moments the ones given by* (12) *is the p.d.f*

$$f^*(x) = \frac{1}{2\sqrt{3m}} \quad \text{for } |x| < \sqrt{3m}. \tag{13}$$

Proof. Since

$$\limsup \frac{(\mu^*_n)^{1/n}}{n} = 0$$

we conclude that the moments m^*_{2n} determine uniquely the distribution (Feller [3]). The characteristic function is

$$\Psi(t) = \sum_{n=0}^{\infty} \frac{m^*_{2n}(it)^{2n}}{(2n)!}.$$

Utilizing (12), the characteristic function takes the form

$$\begin{aligned}\Psi(t) &= \sum_{n=0}^{\infty} \frac{(\sqrt{3m}it)^{2n}}{(2n+1)!}\\ &= \frac{1}{\sqrt{3m}it} \sum_{n=0}^{\infty} \frac{(\sqrt{3m}it)^{2n+1}}{(2n+1)!}\\ &= \frac{1}{\sqrt{3m}t} \sin(\sqrt{3m}t).\end{aligned}$$

This characteristic function corresponds to the random variable $X = \sqrt{3m}Y$ where $Y \sim U(-1,1)$. In other words the p.d.f. f^* is given by (13). $\diamondsuit$

Let us define a narrower class $\Psi_m^{(2)}$ of p.d.f's imposing the extra condition

P6. The p.d.f is unimodal.

In other words

$$\Psi_m^{(2)} = \{f : f \text{ is a p.d.f. which satisfies the properties P1 through P6}\}.$$

In this case the bounds from Proposition 1 hold for $\Psi_m^{(2)}$ and they can not be improved. Indeed f^* does not belong to the set $\Psi_m^{(2)}$ because it is not unimodal. However we can prove the following:

Lemma 3. *The p.d.f. f^* belongs to the boundary of the set $\Psi_m^{(2)}$.*

Proof. Let us define the class of p.d.f.'s from $\Psi_m^{(2)}$

$$g(x,\alpha,r) = \begin{cases} \frac{(2r-1)}{a^2}x + \frac{1-r}{\alpha} & 0 \le x \le a \\ \frac{1-2r}{\alpha^2}x + \frac{1-r}{a} & -a \le x \le 0 \end{cases}$$

where $r \le 1/2$ and $\alpha, r > 0$.

We can easily prove that the moments of order $2n$ are given by

$$m_{2n}(g) = \alpha^{2n}\frac{(2rn+1)}{(n+1)(2n+1)}.$$

For the second moment we have

$$m_2(g) = \alpha^2\frac{2r+1}{6}.$$

Apparently for every $r \leq 1/2$ we can find α_r such that

$$m_2(g) = \alpha_r^2 \frac{2r+1}{6} = m. \tag{14}$$

From (14) we obtain that $\alpha_r^2 \to 3m$ as $r \to 1/2$. Furthermore for the p.d.f.'s $g_r(x) = g(x, \alpha_r, r)$ we have that

$$g_r(x) \to f^*(x) \text{ for every } x < \sqrt{3m},$$

as well as

$$m_{2k}(g_r) \to m_{2k}^*. \tag{12}$$

$\diamondsuit$

Because of property P3, the p.d.f's with countable support are not included in $\Psi_m^{(1)}$ and $\Psi_m^{(2)}$. If we try to find lower bounds for the wider class of p.d.f.'s $\Psi_m^{(3)}$ which includes all the p.d.f.'s with all absolute moments ν_k finite, and the same $\nu_1 = m$, we must use the inequality (8).

Working as in Proposition 1, we obtain

$$\nu_k \geq m^k. \tag{14}$$

In this case the lower bounds m^k $k = 1, 2, \ldots$, are the absolute moments of the degenerate r.v. with p.f.d.

$$f(x) = \begin{cases} 1 & x = m \\ 0 & x \neq m. \end{cases}$$

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